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On integrals over $(0, \infty)$ of functions of the type $f(t) = \exp(-x\phi(t))-1$

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On integrals over $(0,\infty)$ of functions of the type $f(t) = \exp(-x\varphi(t)) - 1$

by C.G. Lekkerkerker

1. Recently, I was told by my colleague Mr A.H.M. Levelt that, in a physical discussion on the attraction potential of atoms [1], the following question arose.

Problem. Let $\varphi(t)$ be a real, continuous function on the interval $[0,\infty)$. Suppose that the integral

(1) $$I(x) = \int_0^\infty (e^{-x\varphi(t)} - 1)\,dt$$

converges for all $x > 0$. In how far the function $\varphi(t)$ is then determined by the function $I(x)$?

In the following we shall answer this question. First, we shall show, at hand of a simple example, that the integral in the right hand member of (1) does not necessarily converge absolutely.

Theorem 1. Let $\varphi(t)$ be continuous on $[0,\infty)$ and let $\varphi^*(t)$ be defined by

(2) $$\varphi^*(t) = \begin{cases} \varphi(t) & \text{if } |\varphi(t)| \leq 1 \\ 0 & \text{if } |\varphi(t)| > 1 . \end{cases}$$

Suppose that the integral (1) converges for all $x > 0$. Then the integral

(3) $$I^*(x) = \int_0^\infty \left\{ e^{-x\varphi(t)} - 1 + x\varphi^*(t) \right\} dt$$

converges absolutely for $x > 0$. This result remains true if one takes $\varphi^*(t) = \operatorname{sign}\varphi(t)$ for $|\varphi(t)| > 1$.

This theorem will enable us to treat the problem stated by making use of known results (viz. the inversion formula) for absolutely convergent, two-sided Laplace integrals. We shall find that there is a large class of functions $\varphi(t)$ leading to the same function $I(x)$. In the case that $\varphi(t)$ is positive (so that the integral (1) converges absolutely) this class, which will be characterized explicitly, contains exactly one monotonic function (see theorem 2 and the final remarks).

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1) This discussion took place at a colloquium in the "Van der Waals Laboratorium", Municipal University of Amsterdam.

2. The example meant above is as follows. Let n run through the positive integers and consider the function $\psi(t)$ defined by

$$\psi(t) = \begin{cases} 1 & \text{for } 0 \leqq t < 1 \\ t^{-2/3} & \text{for } 2n \leqq t < 2n+1 \\ -t^{-2/3} & \text{for } 2n-1 \leqq t < 2n \,. \end{cases}$$

One has

$$\int_1^a (e^{-x\psi(t)}-1)dt = -x \int_1^a \{ \psi(t) + O(t^{-4/3}) \} dt$$

$$= -x \sum_{k=1}^{[a]} (-1)^k \quad k^{-2/3} + O(\int_1^a t^{-4/3} dt),$$

and so $\int^{\infty} (e^{-x\psi(t)}-1)dt$ converges. On the other hand, $\int_0^{\infty} |e^{-x} \psi^0(t)_{-1}|dt \sim -\int_0^{\infty} (e^{-x|\psi(t)|}-1)dt$ clearly diverges. It is easy to construct a function $\varphi(t)$, which has the same properties and, in addition, is continuous on $[0,\infty)$.

In the following a fundamental role is played by the function $\mu(u)$ defined by

$$(4) \qquad \mu(u) = \begin{cases} \mu \{ t | \varphi(t) > u \} & \text{if } u \geqq 0 \\ -\mu \{ t | \varphi(t) < u \} & \text{if } u < 0 \end{cases} ,$$

where on the right the Lebesgue measure of the indicated set of numbers $t > 0$ is meant. We also introduce, for arbitrary $a > 0$, the functions $\varphi_a(t)$ $(t \geqq 0)$ and $\mu_a(u)$ given by

$$(5) \qquad \varphi_a(t) = \begin{cases} \varphi(t) & \text{for } 0 \leqq t \leqq a \\ 0 & \text{for } \ t > a \end{cases}$$

$$(6) \qquad \mu_a(u) = \begin{cases} \mu \{ t \mid \varphi_a(t) > u \} & \text{if } u \geqq 0 \\ -\mu \{ t \mid \varphi_a(t) < u \} & \text{if } u < 0 \end{cases} .$$

We wish to express the integrals (1) and (3) as integrals depending on the function $\mu(u)$. This is done as follows. Let $a > 0$ be arbitrary. Then $\mu_a(u)$ vanishes if $|u|$ is sufficiently large. Further, $\mu_a(u)$ is bounded and, if u, u' are of the same sign and $u' > u$,

$$\mu_a(u') - \mu_a(u) = -\mu \{ t | \ u < \varphi(t) \leqq u' \} .$$

Then, by well-known arguments, since $e^{-xu}-1=0$ for $u=0$ [2],

$$\int_0^a (e^{-x\varphi_a(t)}-1)dt = -\int_{-\infty}^{\infty}(e^{-xu}-1)d\mu_a(u).$$

Since $\mu_a(u)$ vanishes for $|u|$ sufficiently large, partial integration yields

$$(7)\qquad \int_0^a (e^{-x\varphi(t)}-1)dt = -x\int_{-\infty}^{\infty} e^{-xu}\mu_a(u)du \quad (a>0).$$

Now suppose that <u>the integral (1) converges absolutely</u> (or, what comes to the same thing, that the integrals of $e^{-xu}\mu(u)$ over $(0,\infty)$ and $(-\infty,0)$ are finite), and, in the last formula, pass to the limit for $a\to\infty$. Since $\mu_a(u)$ is positive and a non-decreasing function of a if $u>0$ and $-\mu_a(u)$ is likewise positive and non-decreasing if $u<0$, one has [3]

$$\lim_{a\to\infty}\int_0^a (e^{-x\varphi(t)}-1)dt = -x\int_{-\infty}^{\infty} e^{-xu}\lim_{a\to\infty}\mu_a(u)du,$$

and so one gets

$$(8)\qquad I(x) = \int_0^{\infty}(e^{-x\varphi(t)}-1)dt = -x\int_{-\infty}^{\infty} e^{-xu}\mu(u)du.$$

It follows from our deduction that, under the hypothesis made, the function $\mu(u)$ is finite for $u\neq 0$.

In a similar way one can derive that

$$(9)\qquad I^*(x) = \int_0^{\infty}\left\{e^{-x\varphi(t)}-1+x\varphi^*(t)\right\}dt$$

$$= -x\int_{-1}^{1}(e^{-xu}-1)\mu(u)du - x\int_{|u|\geq 1} e^{-xu}\mu(u)du,$$

provided that the first integral converges absolutely or that the integrals in the last member are finite.

Next, we come to the

<u>Proof of theorem 1</u>. Let x be any positive number. Consider the two integrals

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2) An approximating sum to the Stieltjes integral in the formula stated is also an approximating sum to the integral on the left considered as a Lebesgue integral.

3) See E.C. Titchmarsh, Theory of functions, Oxford 1939, theorem 10.82.

$$\int_{-\infty}^{\infty} e^{-xu}\mu_a(u)du, \quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}xu}\mu_a(u)du \qquad (a>0).$$

In virtue of the relation (7) and the conditions of the theorem these two integrals are bounded in absolute value by a constant $c=c(x)$ not depending on a. Further, since $\mu_a(u)$ is $\geqq 0$ for $u>0$ and $\leqq 0$ for $u<0$,

$$e^{-xu}\mu_a(u) \leqq e^{-\frac{1}{2}xu}\mu_a(u)$$

for $u>0$ as well as for $u<0$. Hence,

$$\int_{-\infty}^{\infty}(e^{-\frac{1}{2}xu}-e^{-xu})\mu_a(u)du \leqq 2c(x),$$

where the integrand is nonnegative throughout the interval $(-\infty,\infty)$. Since $\mu(u)=\lim_{a\to\infty}\mu_a(u)$, we also have [4]

$$\int_{-\infty}^{\infty}(e^{-\frac{1}{2}xu}-e^{-xu})\mu(u)du \leqq 2c(x).$$

Then, since

$$\left| e^{-\frac{1}{2}xu}-e^{-xu} \right| \geqq \begin{cases} (1-e^{-\frac{1}{2}x})e^{-\frac{1}{2}xu} & \text{if } u \geqq 1 \\ e^{-\frac{1}{2}x}\left| e^{-\frac{1}{2}xu}-1\right| & \text{if } |u| \leqq 1 \\ (e^{\frac{1}{2}x}-1)e^{-\frac{1}{2}xu} & \text{if } u \leqq -1 \end{cases},$$

the integrals

$$\int_{1}^{\infty} e^{-\frac{1}{2}xu}\mu(u)du, \quad \int_{-\infty}^{-1} e^{-\frac{1}{2}xu}\mu(u)du, \quad \int_{-1}^{1} (e^{-\frac{1}{2}xu}-1)\mu(u)du$$

are all finite. This means that the integral $I^*(\frac{1}{2}x)$ converges absolutely (see formula (9)).

Since $x>0$ was arbitrary, this proves the first assertion of the theorem. Since $\mu(1)$ and $\mu(-1)$ are finite, the second assertion also holds.

3. We now state and prove the following

Theorem 2. Let $\varphi(t)$ and $\psi(t)$ be two functions which are continuous on $[0,\infty)$, and suppose that the integrals

$$I(x) = \int_{0}^{\infty}(e^{-x\varphi(t)}-1)dt, \quad I_1(x) = \int_{0}^{\infty}(e^{-x\psi(t)}-1)dt$$

converge for $x>0$. Let $\mu(u)$ be given by (4), and let $\nu(u)$ be defined similarly, with $\psi(t)$ instead of $\varphi(t)$.

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4) See footnote 3).

Then the functions $I(x)$ and $I_1(x)$ are identical if and only if the functions $\mu(u)$ and $\psi(u)$ are identical.

Proof. We first consider the case that the integrals $I(x)$ and $I_1(x)$ are absolutely convergent for $x > 0$. Then for $I(x)$ formula (8) holds. Applying the inversion formula for absolutely convergent, two-sided Laplace integrals we get

$$\mu(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} I(x)\, e^{ux}\, dx \qquad (c > 0).$$

Similar formulae hold for $I_1(x)$ and $\nu(u)$.
From this the assertion of the theorem follows.

Next, we deal with the more general case, in which the integrals $I(x)$ and $I_1(x)$ do not necessarily be absolutely convergent. Then, at any rate, in virtue of theorem 1 the integral $I^*(x)$ is absolutely convergent for $x > 0$. Further, formula (9) holds. Similar remarks hold for the integral $I_1^*(x)$ obtained from $I^*(x)$ by replacing $\varphi(t)$ by $\psi(t)$. We note that from these facts and the conditions of the theorem it follows that the integrals $\int_0^\infty \varphi(t)dt$ and $\int_0^\infty \psi(t)dt$ converge and that

$$(10) \qquad I^*(x) - I_1^*(x) = I(x) - I_1(x) + \beta x ,$$

where $\beta = \int_0^\infty \{\varphi^*(t) - \psi^*(t)\}\, dt$ is a finite constant.

We introduce functions $\mu_1(u)$ and $\nu_1(u)$ as follows:

$$\begin{matrix}\mu_1(u)\\ \nu_1(u)\end{matrix} = \int_u^1 \begin{matrix}\mu(u)\\ \nu(u)\end{matrix}\, du \quad \text{or} \quad \int_u^{-1} \begin{matrix}\mu(u)\\ \nu(u)\end{matrix}\, du ,$$

according as to whether $u > 0$ or $u < 0$. In virtue of (9) the integral $\int^1 u\mu(u)du$ is finite. Then $u\mu_1(u)$ tends to zero if $u \to +0$ [5], and also if $u \to -0$. Similarly, $u\nu_1(u)$ tends to zero if $u \to +0$ or -0. Further, $e^{-xu}\mu_1(u)$ and $e^{-xu}\nu_1(u)$ tend to zero for $u \to \infty$ and for $u \to -\infty$. Hence, by partial integration, we find

$$\begin{aligned} I^*(x) &= -x \int_{-1}^{1} (e^{-xu}-1)\mu(u)du - x \int_{|u|\geq 1} e^{-xu}\mu(u)du \\ &= -x^2 \int_{-1}^{1} e^{-xu}\mu_1(u)du - x^2 \int_{|u|\geq 1} e^{-xu}\mu_1(u)du \end{aligned}$$

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5) If $\varepsilon > 0$ is chosen arbitrarily, then $\int_\delta^{\delta_1} u\mu(u)du < \varepsilon$ for a suitably chosen δ_1 and $0 < \delta < \delta_1$, hence
$\delta\mu_1(\delta) = \delta \int_\delta^1 \mu(u)du < \int_\delta^{\delta_1} u\mu(u)du + \delta \int_{\delta_1}^1 \mu(u)du < 2\varepsilon$
for sufficiently small δ.

$$= -x^2 \int_{-\infty}^{\infty} e^{-xu} \mu_1(u)\, du \;,$$

and similarly

$$\overset{*}{I}_1(x) = -x^2 \int_{-\infty}^{\infty} e^{-xu}\, \nu_1(u) du.$$

It follows from these results and the relation (10) that $I(x)$ and $I_1(x)$ are identical, if and only if

$$\mu_1(u) - \nu_1(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{-\beta}{x} e^{ux}\, dx = -\beta \;;$$

here necessarily $\beta = 0$, because of $\mu_1(1) = \nu_1(1) = 0$. This proves the theorem.

<u>Final remarks.</u> It is easy to construct two continuous functions $\varphi(t)$ and $\psi(t)$, leading to identical functions $I(x)$ and $I_1(x)$. Let $\varphi(t)$ be continuous on $[0,\infty)$ and suppose that $\varphi(0) = \varphi(1) = \varphi(2)$. Further, take

$$\psi(t) = \begin{cases} \varphi(t+1) & \text{for } 0 \leqq t < 1 \\ \varphi(t-1) & \text{for } 1 \leqq t < 2 \\ \varphi(t) & \text{for } \quad t \geqq 2 \;. \end{cases}$$

Then clearly $\int_0^{\infty} (e^{-x\,\varphi(t)} - 1) dt = \int_0^{\infty} (e^{-x\,\psi(t)} - 1) dt$, if the first integral converges.

In general, the last relation holds, if the first integral converges and $\psi(t) = \varphi(Lt)$, where L is any one-to-one, measure preserving mapping of $[0,\infty)$ onto itself, such that $\int_0^{\infty} \varphi(t) dt$ is unchanged. If $\varphi(t)$ is positive, there is exactly one such mapping, for which $\varphi(Lt)$ is monotonic (and continuous); actually, $\varphi(Lt)$ is the inverse function of $t = \mu(u)$.

In the physical discussion meant in the introduction the function $\varphi(t)$ was monotoneously decreasing from $+\infty$ to some negative value on some interval $[0,t_0)$ and monotoneously increasing to 0 on the interval (t_0,∞). Here we get the same integral $I(x)$, if we apply any deformation to the graph of $u = \varphi(t)$, such that the lengths of the horizontal line-segments with endpoints on this graph remain unchanged and such that we get the graph of some function $u = \psi(t)$.